

OSCILLATION CRITERIA FOR FIRST AND SECOND-ORDER DIFFERENCE EQUATIONS

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ABSTRACT

Consider the first-order and the second-order delay difference equations

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)$$

and

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (2)$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \rightarrow \mathbb{R}^+$, $\tau : \mathbb{N} \rightarrow \mathbb{N}$, $\tau(n) \leq n-1$ and $\lim_{n \rightarrow \infty} \tau(n) = +\infty$,

The most interesting oscillation criteria for Eq.(1), and Eq. (2), especially in the case where

$$0 < \liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sum_{i=\tau(n)}^n p(i) < 1.$$

for Eq.(1), are presented.

1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions of the first-order delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)$$

has been the subject of many investigations, especially in the case where the delay $n - \tau(n)$ is a constant, that is, in the special case of the difference equation

$$\Delta x(n) + p(n)x(n - k) = 0, \quad n = 0, 1, 2, \dots \quad (1)'$$

⁰*Key Words:* Oscillation; delay, difference, differential equations.

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The oscillation theory of the second-order delay difference equation

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0, \quad (2)$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \rightarrow \mathbb{R}^+$, $\tau : \mathbb{N} \rightarrow \mathbb{N}$, k is a positive integer, $\tau(n) \leq n-1$ and $\lim_{n \rightarrow \infty} \tau(n) = +\infty$, has also attracted growing attention in the recent few years. See, for example, [1, 2, 4-16, 18, 19, 24, 26, 30, 36, 40, 42, 43, 45-47, 52-55, 58-64, 68, 69, 71-76] and the references cited therein.

Strong interest in Eq.(1); Eq.(1)', and Eq. (2), are motivated by the fact that they represent discrete analogues of the delay differential equations

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1_c)$$

$$x'(t) + p(t)x(t - \tau) = 0, \quad \tau > 0. \quad (1_c)'$$

and

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (2_c)$$

respectively, where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. See [3, 17, 20-23, 25, 27-29, 31-35, 37-39, 41, 44, 48-51, 56, 57, 65-67, 70] and the references cited therein.

By a solution of Eq.(1) we mean a sequence $x(n)$ which is defined for $n \geq \min \{\tau(n) : n \geq 0\}$ and which satisfies Eq.(1) for all $n \geq 0$. A solution $x(n)$ of Eq.(1) is said to be *oscillatory* if the terms $x(n)$ of the solution are neither eventually positive nor eventually negative. Otherwise the solution is called *nonoscillatory*. (Analogously for Eq.(1)' and Eq.(2))

In this paper our purpose is to present the state of the art on the oscillation of all solutions to Eq.(1), Eq. (1)' and Eq. (2), especially in the case where

$$0 < \liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sum_{i=\tau(n)}^n p(i) < 1.$$

for Eq.(1), and

$$0 < \liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) \leq \left(\frac{k}{k+1} \right)^{k+1} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sum_{i=n-k}^n p(i) < 1$$

for Eq.(1)'.

2 Oscillation Criteria for Eq. (1)'

In this section we study the difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, \dots \quad (1)'$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and k is a positive integer.

In 1981, Domshlak [14] was the first who studied this problem in the case where $k = 1$. Then, in 1989, Erbe and Zhang [24] established that all solutions of Eq.(1)' are oscillatory if

$$\liminf_{n \rightarrow \infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \quad (2.1)$$

or

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^n p(i) > 1. \quad (C_1)'$$

In the same year, 1989, Ladas, Philos and Sficas [43] proved that a sufficient condition for all solutions of Eq.(1)' to be oscillatory is that

$$\liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1} \right)^{k+1} \quad (C_2)'$$

Therefore they improved the condition (2.1) by replacing the $p(n)$ of (2.1) by the arithmetic mean of $p(n-k), \dots, p(n-1)$ in $(C_2)'$.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (2.1) it should be emphasized that, as it is shown in [24], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}$$

then Eq.(1)' has a nonoscillatory solution.

In 1990, Ladas [42] conjectured that Eq.(1)' has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) < \left(\frac{k}{k+1} \right)^{k+1}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Yu, Zhang and Wang [73].

It is interesting to establish sufficient oscillation conditions for the equation (1)' in the case where neither $(C_1)'$ nor $(C_2)'$ is satisfied.

In 1995, the following oscillation criterion was established by Stavroulakis [54]:

Theorem 2.1 ([54]) *Assume that*

$$\alpha_0 := \liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) \leq \left(\frac{k}{k+1} \right)^{k+1}$$

and

$$\limsup_{n \rightarrow \infty} p(n) > 1 - \frac{\alpha_0^2}{4} \quad (2.2)$$

then all solutions of Eq.(1)' oscillate.

In 2004, the same author [55] improved the condition (2.2) as follows:

Theorem 2.2 ([55]) *If $0 < \alpha_0 \leq \left(\frac{k}{k+1} \right)^{k+1}$, then either one of the conditions*

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{4} \quad (C_3)'$$

or

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha_0^k \quad (2.3)$$

implies that all solutions of Eq.(1)' oscillate.

In 2006, Chatzarakis and Stavrulakis [8], established the following

Theorem 2.3 ([8]) *If $0 < \alpha_0 \leq \left(\frac{k}{k+1} \right)^{k+1}$ and*

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{2(2 - \alpha_0)} \quad (2.4)$$

then all solutions of Eq.(1)' oscillate.

Remark 2.1. Observe the following:

(i) When $\alpha \rightarrow 0$, then it is clear that the conditions $(C_3)'$, (2.3) and (2.4) reduce to

$$A := \limsup_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} p_i > 1,$$

which obviously implies $(C_1)'$.

(ii) It always holds

$$\frac{\alpha^2}{2(2 - \alpha)} > \frac{\alpha^2}{4},$$

since $\alpha > 0$ and therefore condition $(C_3)'$ always implies (2.4).

(iii) When $k = 1, 2$

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^k,$$

(since, from the above mentioned conditions, it makes sense to investigate the case when $\alpha \leq \left(\frac{k}{k+1}\right)^{k+1}$) and therefore condition (2.4) implies (2.3).

(iv) When $k = 3$,

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^3 \text{ if } 0 < \alpha < 1 - \frac{\sqrt{2}}{2}$$

while

$$\frac{\alpha^2}{2(2-\alpha)} < \alpha^3 \text{ if } 1 - \frac{\sqrt{2}}{2} < \alpha \leq \left(\frac{3}{4}\right)^4.$$

So in this case the conditions (2.4) and (2.3) are independent.

(v) When $k \geq 4$

$$\frac{\alpha^2}{2(2-\alpha)} > \alpha^k,$$

and therefore condition (2.3) implies (2.4).

(vi) When $k \geq 10$ condition (2.4) may hold but condition $(C_1)'$ may not hold.

(vii) When k is large then $\alpha \rightarrow \frac{1}{e}$ and in this case both conditions $(C_3)'$ and (2.3) imply (2.4). For illustrative purposes, we give the values of the lower bound of A under these conditions when $k = 100$ ($\alpha \simeq 0.366$):

$$(2.3) : 0.999999$$

$$(C_3)' : 0.966511$$

$$(2.4) : 0.959009$$

We see that the condition (2.4) essentially improves the conditions $(C_3)'$ and (2.3).

Also, Chen and Yu [9] obtained the following oscillation condition

$$\limsup_{n \rightarrow \infty} \sum_{i=n-k}^n p(i) > 1 - \frac{1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}}{2}. \quad (C_4)'$$

3 Oscillation Criteria for Eq. (1)

In this section we study the difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, \dots, \quad (1)$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n \rightarrow \infty} \tau(n) = \infty$.

In the case of Eq.(1) with a general delay argument $\tau(n)$, from Chatzarakis, Koplatadze and Stavroulakis [4], it follows the following

Theorem 3.1 ([4]) *If*

$$\limsup_{n \rightarrow \infty} \sum_{i=\tau(n)}^n p(i) > 1 \quad (C_1)$$

then all solutions of Eq. (1) oscillate.

This result generalizes the oscillation criterion $(C_1)'$. Also Chatzarakis, Koplatadze and Stavroulakis [5] extended the oscillation criterion $(C_2)'$ to the general case of Eq. (1). More precisely, the following theorem has been established in [5].

Theorem 3.2 ([5]) *Assume that*

$$\limsup_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty \quad (3.1)$$

and

$$\alpha := \liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e}. \quad (C_2)$$

Then all solutions of Eq.(1) oscillate.

Remark 3.1 It is to be pointed out that the conditions (C_1) and (C_2) are the discrete analogues of the conditions $(C_1)'$ and $(C_2)'$ for Eq.(1) in the case of a general delay argument $\tau(n)$.

Remark 3.2 ([5]). The condition (C_2) is optimal for Eq.(1) under the assumption that $\lim_{n \rightarrow +\infty} (n - \tau(n)) = \infty$, since in this case the set of natural numbers increases infinitely in the interval $[\tau(n), n-1]$ for $n \rightarrow \infty$.

Now, we are going to present an example to show that the condition (C_2) is optimal, in the sense that it cannot be replaced by the non-strong inequality.

Example 3.1 ([5]) Consider Eq.(1), where

$$\tau(n) = [\beta n], \quad p(n) = (n^{-\lambda} - (n+1)^{-\lambda}) ([\beta n])^\lambda, \quad \beta \in (0, 1), \quad \lambda = -\ln^{-1} \beta \quad (3.2)$$

and $[\beta n]$ denotes the integer part of βn .

It is obvious that

$$n^{1+\lambda} (n^{-\lambda} - (n+1)^{-\lambda}) \rightarrow \lambda \quad \text{for } n \rightarrow \infty.$$

Therefore

$$n (n^{-\lambda} - (n+1)^{-\lambda}) ([\beta n])^\lambda \rightarrow \frac{\lambda}{e} \quad \text{for } n \rightarrow \infty. \quad (3.3)$$

Hence, in view of (3.2) and (3.3), we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) &= \frac{\lambda}{e} \liminf_{n \rightarrow \infty} \sum_{i=[\beta n]}^{n-1} \frac{e}{\lambda} i (i^{-\lambda} - (i+1)^{-\lambda}) ([\beta i])^\lambda \cdot \frac{1}{i} \\ &= \frac{\lambda}{e} \liminf_{n \rightarrow \infty} \sum_{i=[\beta n]}^{n-1} \frac{1}{i} = \frac{\lambda}{e} \ln \frac{1}{\beta} = \frac{1}{e} \end{aligned}$$

or

$$\liminf_{n \rightarrow \infty} \sum_{i=\tau(n)}^{n-1} p(i) = \frac{1}{e}. \quad (3.4)$$

Observe that all the conditions of Theorem 3.2 are satisfied except the condition (C_2) . In this case it is not guaranteed that all solutions of Eq.(1) oscillate. Indeed, it is easy to see that the function $u = n^{-\lambda}$ is a positive solution of Eq.(1).

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference equation (1), in the case where neither (C_1) nor (C_2) is satisfied.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [4] investigated for the first time this question for Eq.(1) in the case of a general delay argument $\tau(n)$ and derived the following theorem.

Theorem 3.3 ([4]) *Assume that $0 < \alpha \leq \frac{1}{e}$. Then we have:*

(I) *If*

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 - (1 - \sqrt{1 - \alpha})^2 \quad (3.5)$$

then all solutions of Eq.(1) oscillate.

(II) *If in addition,*

$$p(n) \geq 1 - \sqrt{1 - \alpha} \quad \text{for all large } n, \quad (3.6)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 - \alpha \frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}} \quad (3.7)$$

then all solutions of Eq.(1) oscillate.

Recently, the above result was improved in [6] and [7] as follows:

Theorem 3.4 ([6]) (I) If $0 < \alpha \leq \frac{1}{e}$ and

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 - \frac{1}{2} (1 - \alpha - \sqrt{1 - 2\alpha}) \quad (3.8)$$

then all solutions of Eq.(1) oscillate.

(II) If $0 < \alpha \leq 6 - 4\sqrt{2}$ and in addition,

$$p(n) \geq \frac{\alpha}{2} \text{ for all large } n, \quad (3.9)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 - \frac{1}{4} (2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}) \quad (3.10)$$

then all solutions of Eq.(1) are oscillatory.

Theorem 3.5 ([7]) Assume that $0 < \alpha \leq -1 + \sqrt{2}$, and

$$\limsup_{n \rightarrow \infty} \sum_{j=\tau(n)}^n p(j) > 1 - \frac{1}{2} (1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}) \quad (C_4)$$

then all solutions of Eq.(1) oscillate.

Remark 3.3 Observe the following:

(i) When $0 < \alpha \leq \frac{1}{e}$, it is easy to verify that

$$\frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} > \alpha \frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}} > \frac{1 - \alpha - \sqrt{1 - 2\alpha}}{2} > (1 - \sqrt{1 - \alpha})^2$$

and therefore the condition (C_4) is weaker than the conditions (3.7), (3.8) and (3.5).

(ii) When $0 < \alpha \leq 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4} (2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}) > \frac{1}{2} (1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}),$$

and therefore in this case and when (3.9) holds, inequality (3.10) improves the inequality (C_4) and especially, when $\alpha = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in (C_4) is 0.8929094 while in (3.10) is 0.7573593.

4 Oscillation Criteria for Eq. (2)

In this section we study the second-order difference equation

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0 \quad (2)$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \rightarrow \mathbb{R}_+$, $\tau : \mathbb{N} \rightarrow \mathbb{N}$, $\tau(n) \leq n-1$ and $\lim_{n \rightarrow \infty} \tau(n) = +\infty$.

In 1994, Wyrwiska [69] proved that all solutions of Eq. (2) are oscillatory if

$$\limsup_{n \rightarrow \infty} \left\{ \sum_{i=\tau(n)}^n [\tau(i) - 2]p(i) + [\tau(n) - 2] \sum_{i=n+1}^{\infty} p(i) \right\} > 1,$$

while, in 1997, Agarwal, Thandapani and Wong [1] proved that, in the special case of the second-order difference equation with constant delay

$$\Delta^2 x(n) + p(n)x(n-k) = 0, \quad k \geq 1 \quad (2_c)'$$

all solutions are oscillatory if

$$\liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} (i-k)p(i) > 2 \left(\frac{k}{k+1} \right)^{k+1}.$$

In 2001, Grzeczorczyk and Werbowksi [26] studied Eq.(2_c)' and proved that under the following conditions

$$\limsup_{n \rightarrow \infty} \left\{ \frac{\sum_{i=n-k}^n (i-n+k+1)p(i) + \left[(n-k-2) + \sum_{i=n_1}^{n-k-1} (i-k)^2 p(i) \right] \times \sum_{i=n+1}^{\infty} p(i)}{\sum_{i=n+1}^{\infty} p(i)} \right\} > 1, \text{ for some } n_1 > n_0,$$

or

$$\liminf_{n \rightarrow \infty} \sum_{i=n-k}^{n-1} (i-k-1)p(i) > \left(\frac{k}{k+1} \right)^{k+1} \quad (C_2)''$$

all solutions of Eq. (2_c)' are oscillatory. Observe that the last condition (C₂)'' may be seen as the discrete analogue of the condition

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t \tau(s)p(s)ds > \frac{1}{e}$$

for Eq. (2_c).

In 2001 Koplatadze [36] studied the oscillatory behaviour of all solutions to the equation (2) with variable delay and established the following.

Theorem 4.1 ([36]) *Assume that*

$$\inf \left\{ \frac{1}{1-\lambda} \liminf_{n \rightarrow \infty} n^{-\lambda} \sum_{i=1}^n ip(i)\tau^\lambda(i) : \lambda \in (0, 1) \right\} > 1$$

and

$$\liminf_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n ip(i)\tau(i) > 0.$$

Then all solutions of Eq.(2) oscillate.

Corollary 4.1 ([36]) *Let $\alpha > 0$ and*

$$\liminf_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n i^2 p(i) > \max \{ \alpha^{-\lambda} \lambda (1 - \lambda) : \lambda \in [0, 1] \}.$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0, \quad n \geq \max \left\{ 1, \frac{1}{\alpha} \right\}, \quad n \in N$$

oscillate.

Corollary 4.2 ([36]) *Let n_0 be an integer and*

$$\liminf_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n i^2 p(i) > \frac{1}{4}.$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n - n_0) = 0, \quad n \geq \max\{1, n_0 + 1\}, \quad n \in N$$

oscillate.

In 2002 Koplatadze, Kvinikadze and Stavroulakis [40] studied Eq.(2) and established the following.

Theorem 4.2 ([40]) *Assume that*

$$\liminf_{n \rightarrow \infty} \frac{\tau(n)}{n} = \alpha \in (0, \infty),$$

and

$$\liminf_{n \rightarrow \infty} n \sum_{i=n}^{\infty} p(i) > \max \{ \alpha^{-\lambda} \lambda (1 - \lambda) : \lambda \in [0, 1] \}. \quad (4.1)$$

Then all solutions of Eq.(2) oscillate.

In the case where $\alpha = 1$, the following discrete analogue of Hille's well-known oscillation theorem for 2nd order ordinary differential equations (see [29]) is derived.

Theorem 4.3 ([40]) *Let n_0 be an integer and*

$$\liminf_{n \rightarrow \infty} n \sum_{i=n}^{\infty} p(i) > \frac{1}{4}. \quad (4.2)$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n - n_0) = 0, \quad n \geq n_0,$$

oscillate.

Remark 4.1 ([40]) As in case of ordinary differential equations, the constant $1/4$ in (4.2) is optimal in the sense that the strict inequality cannot be replaced by the nonstrict one. More than that, the same is true for the condition (4.1) as well. To ascertain this, denote by c the right-hand side of (4.1), and by λ_0 the point where the maximum is achieved. The sequence $x(n) = n^{\lambda_0}$ obviously is a nonoscillatory solution of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0,$$

where $p(n) = -\Delta^2(n^{\lambda_0})/[\alpha n]^{\lambda_0}$ and $[\alpha]$ denotes the integer part of α . It can be easily calculated that

$$p(n) = -\frac{c}{n^2} + o\left(\frac{1}{n^2}\right) \quad \text{as } n \rightarrow \infty.$$

Hence for arbitrary $\varepsilon > 0$, $p(n) \geq (c - \varepsilon)/n^2$ for $n \in \mathbb{N}_{\kappa\mu}$ with $n_0 \in \mathbb{N}$ sufficiently large. Using the inequality $\sum_{i=n}^{\infty} i^2 \geq n^{-1}$ and the arbitrariness of ε , we obtain

$$\liminf_{n \rightarrow \infty} n \sum_{i=n}^{\infty} p(i) \geq c.$$

This limit can not be greater than c by Theorem 4.2. Therefore it equals c and (4.1) is violated.

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